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Fixed point properties of semigroups of non-expansive mappings

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Abstract

In recent years, there have been considerable interests in the study of when a closed convex subset K of a Banach space has the fixed point property, i.e. whenever T is a non-expansive mapping from K into K, then K contains a fixed point for T. In this paper we shall study fixed point properties of semigroups of non-expansive mappings on weakly compact convex subsets of a Banach space (or, more generally, a locally convex space). By considering the classes of bicyclic semigroups we answer two open questions, one posted earlier by the first author in 1976 (Dalhousie) and the other posted by T. Mitchell in 1984 (Virginia). We also provide a characterization for the existence of a left invariant mean on the space of weakly almost periodic functions on separable semitopological semigroups in terms of fixed point property for non-expansive mappings related to another open problem raised by the first author in 1976.

Keywords: Fixed point property; Non-expansive mapping; Weakly compact convex set; Weakly almost periodic; Reversible semigroup; Invariant mean; Bicyclic semigroup

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1. Introduction

In this paper we shall study fixed point properties of semigroups of non-expansive mappings on weakly compact convex subsets of a Banach space (or, more generally, a locally convex space).

Let E be a Banach space and let K be a non-empty bounded closed convex subset of E. We say that K has the *fixed point property* if for every non-expansive mapping $T : K \to K$ (i.e. $\|Tx - Ty\| \leqslant \|x - y\|$, $x, y \in K$), K contains a fixed point for T.

It follows from Bruck [8] that if E is a Banach space with the *weak fixed point property* (i.e. any weakly compact convex subset of E has the fixed point property), then any weakly compact convex subset K of E has the (common) fixed point property for any commutative semigroup acting on K.

A well-known result of Browder [7] asserts that if E is uniformly convex, then E has the weak fixed point property. Kirk [21] extended this result by showing that if K is a weakly compact subset of E with normal structure, then K has the fixed point property. Other examples of Banach spaces with the weak fixed point property include c_0, ℓ^1, trace class operators on a Hilbert space and the Fourier algebra of a compact group (see [12,14,15,26,27,31,32,34,36,40] and [3,4] for more details). However, as shown by Alspach [1], $L^1[0,1]$ does not have the weak fixed point property.

Let S be a *semitopological semigroup*, i.e. S is a semigroup with Hausdorff topology such that for each $a \in S$, the mappings $s \mapsto sa$ and $s \mapsto as$ from S into S are continuous. S is called *left reversible* if any two closed right ideals of S have non-void intersection, i.e. $\overline{aS} \cap \overline{bS} \neq \emptyset$ for any $a, b \in S$. Let Q be a (fixed) family of continuous semi-norms on a separated locally convex space E which determines the topology of E. We denote the space by (E, Q) or simply by E if there is no confusion. Then an action of S on a subset $K \subseteq E$ is Q-non-expansive if $\rho(s \cdot x - s \cdot y) \leqslant \rho(x - y)$ for all $s \in S$, $x, y \in K$ and $\rho \in Q$. The following fixed point property was proved by the first author [22, Theorem 4.1] (see also [37,41]).

Theorem 1.1. *Let S be a semitopological semigroup. Then $AP(S)$, the space of continuous almost periodic functions on S, has a LIM* (*left invariant mean*) *if and only if S has the following fixed point property*:

(D) *Whenever S is a separately continuous and Q-nonexpansive action on a compact convex subset K of a separately locally convex space E, K has a common fixed point for S.*

It has been an open question for quite long time (see [23,25]) as whether the existence of LIM on $WAP(S)$, the space of continuous weakly almost periodic functions on S, can be characterized by a fixed point property for non-expansive actions of S on a weakly compact convex set.

It was proved by Hsu [19] (also see [29, Corollary 5.5]) that if S is discrete and left reversible, then S has the following fixed point property:

(G) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) and the action is weakly separately continuous and Q-non-expansive, then K contains a common fixed point for S.

Since the fixed point property (G) implies that $WAP(S)$ has LIM, it follows that if S is discrete and left reversible, then $WAP(S)$ has a LIM. This improved an earlier result of Ryll-Nardzewski who proved, using his fixed point theorem for affine maps on weakly compact convex subsets of a Banach space, the existence of LIM on $WAP(S)$ when S is a group (see [16]).

It is an open problem (see [23, Problem 5]) whether the existence of LIM on $WAP(S)$ implies fixed point property (G). It is also an open problem for a discrete semigroup S whether the existence of LIM on $WAP(S)$ implies S being left reversible (see [24, Problem 27]).

We shall prove in Section 3 of this paper that, if S is separable, the existence of LIM on $WAP(S)$ can be characterized by a fixed point property (F) of which S is regarded as a semigroup of non-expansive mappings on a weakly compact set. Fixed point property (F) is the same as (G) with the additional assumption that the closure of S (as a set of self-mappings on K) in the topology of pointwise convergence on K consists of weakly continuous maps. In Section 4, we shall study amenability of the class of bicyclic semigroups and use this to show (Theorem 4.11) that there is a semigroup S such that $WAP(S)$ has a LIM (hence has fixed property (F)) but S is not left reversible. This answers a question (see [24, Problem 27]) raised by T. Mitchell in [38]. We also give an example (Theorem 4.13) of a discrete semigroup S that has fixed point property (D) but not fixed point property (F), i.e. $AP(S)$ has LIM but $WAP(S)$ does not have LIM, answering Problem 1 of [23]. In Section 5 we shall consider fixed point properties for semigroups of non-expansive mappings of jointly continuous actions.

It should be noted that property (G) fails without the condition of weak continuity on the action even when the semigroup is commutative. Indeed, Alspach [1] showed that there is a weakly compact convex subset K in $L^1[0,1]$ and a non-expansive map $T: K \to K$ without a fixed point; consequently, if $S = \{T^n\colon\ n = 1, 2, \ldots\}$, then S is a commutative semigroup of non-expansive mappings from K into K without a common fixed point. On the other hand, Belluce and Kirk [2] proved that if K is a non-empty weakly compact convex subset of a Banach space and if K has complete normal structure, then every family of commuting non-expansive self-maps on K has a common fixed point. Later, Lim [33, Theorem 3] extended this theorem to a continuous representation of a left reversible semitopological semigroup S as non-expansive mappings on a weakly compact convex set K with normal structure. In [30], Lau and Takahashi showed that Lim's results remain valid when $CB(S)$, the C*-algebras of bounded complex-valued functions on S, has a left invariant mean. This answered a problem posed during the Conference on Fixed Point Theorem and Applications held at CIRM, Marseille-Luminy, 1989 (see [25, Problem 5, p. 307]).

There is a strong connection between amenability and fixed point properties (see, e.g., [9,25, 28]). Fixed point property (D) was proved for commutative semigroups by De Marr [10], for discrete left amenable semigroups by Takahashi [41], and for discrete left reversible semigroups by Mitchell [37].

It was shown in [29] that if S is left reversible, then $LUC(S)$ has a left invariant (nonlinear) submean. The converse is also true when S is discrete. Using implicitly the notion of invariant submean for a group, Despic and Ghahramani gave in [11] a simple proof of a result of B.E. Johnson [20] on weak amenability of group algebras of a locally compact group. Earlier, using the Ryll-Nardzewski fixed point theorem (see [16]) Yeadon [42] gave a simple proof of the existence of a trace on a finite von Neumann algebra. A recent application of the existence of LIM on $WAP(S)$ when S is a group together with fixed point property (F_2) in [23, p. 123] can be found in the solution of the long standing derivation problem for group algebras by Losert [35].

2. Preliminaries and notations

Throughout this paper, S will denote a semitopological semigroup. Let $\ell^\infty(S)$ be the C*-algebra of bounded complex-valued functions on S with the supremum norm and pointwise multiplication. For each $a \in S$ and $f \in \ell^\infty(S)$ let $\ell_a f$ and $r_a f$ be the left and right translates of f by a, respectively; i.e. $\ell_a f(s) = f(as)$ and $r_a f(s) = f(sa)$ $(s \in S)$. Let X be a closed subspace of $\ell^\infty(S)$ containing constants and be invariant under translations. Then a linear functional $m \in X^*$ is called a *mean* if $\|m\| = m(1) = 1$; m is called a *left* (*respectively right*) *invariant mean*, denoted by LIM (respectively RIM), if $m(\ell_a f) = m(f)$ (respectively $m(r_a f) = m(f)$) for all $a \in S$, $f \in X$. S is *left* (*respectively right*) *amenable* if $\ell^\infty(S)$ has a LIM (respectively RIM). Let X be a C*-subalgebra of $\ell^\infty(S)$. Then the *spectrum* of X is the set of non-zero multiplicative linear functionals on X equipped with the relative weak* topology.

Let $C(S)$ be the space of all bounded continuous complex-valued functions on S. Denote by $AP(S)$ the space of all $f \in C(S)$ such that $\mathcal{LO}(f) = \{\ell_s f\colon s \in S\}$ is relatively compact in the norm topology of $C(S)$, and denote by $WAP(S)$ the space of all $f \in C(S)$ such that $\mathcal{LO}(f)$ is relatively compact in the weak topology of $C(S)$. Functions in $AP(S)$ (respectively $WAP(S)$) are called almost periodic (respectively weakly almost periodic) functions on S. Later in this paper we will also need to consider the set $\mathcal{RO}(f) = \{r_s f\colon s \in S\}$. As well known, $f \in AP(S)$ (respectively $f \in WAP(S)$) if and only if $\mathcal{RO}(f)$ is relatively compact in the norm (respectively weak) topology of $C(S)$. Let S^{a} (respectively S^{w}) be the almost periodic (respectively weakly almost periodic) compactification of S, i.e. S^{a} (respectively S^{w}) is the spectrum of the C*-algebra $AP(S)$ (respectively $WAP(S)$). Then S^{a} and S^{w} are semitopological semigroups with multiplications defined by: $\langle m \cdot n, f\rangle = \langle m, n \cdot f\rangle$, where $n \cdot f(s) = \langle n, \ell_s f\rangle$, $m, n \in S^{\mathrm{a}}$ (respectively S^{w}), $f \in AP(G)$ (respectively $WAP(G)$). In fact, the multiplication in S^{a} is even jointly continuous. In other words, S^{a} is a topological semigroup.

It is known that if S is discrete and left amenable, then S is left reversible. However a general semitopological semigroup S needs not be left reversible even when $C(S)$ has a LIM unless S is normal (see [18]).

When S is a discrete semigroup, the following implication diagram is known [23]:

$$
\begin{array}{ccc}
S \text{ left amenable} & & \\
\Downarrow \;\; \not\Uparrow & & \\
S \text{ left reversible} & & \\
 & \searrow & \\
\Downarrow \;\; \not\Uparrow & & WAP(S) \text{ has LIM} \\
 & \swarrow & \\
AP(S) \text{ has LIM} & &
\end{array}
$$

The implication "S is left reversible $\Rightarrow AP(S)$ has a LIM" for any semitopological semigroup was established in [22]. During the 1984 Richmond, Virginia, conference on analysis on semigroups, T. Mitchell [38] gave two examples to show that for discrete semigroups "$AP(S)$ has LIM" $\not\Rightarrow$ "S is left reversible" (see [24]). The implication "S is left reversible $\Rightarrow WAP(S)$ has LIM" for discrete semigroups was proved by Hsu [19].

If A is a subset of a topological space E, then $\overline{A}$ will denote the closure of A in E. If in addition, E is a linear topological vector space, then $[\overline{\mathrm{co}}\, A]$ $\mathrm{co}\, A$ will denote the [closed] convex hull of A in E.

An action of S on a topological space K is a mapping ψ from $S \times K$ into K, denoted by $sx = \psi(s,x)$ ($s \in S$ and $x \in K$), such that $(s_1s_2)x = s_1(s_2x)$ ($s_1, s_2 \in S$ and $x \in K$). The action is *separately continuous* if the mapping ψ is continuous in each of the variables when the other is kept fixed.

When K is a convex subset of a linear topological space, we say that an action of S on K is affine if for each $s \in S$, the mapping from K into K defined by $x \mapsto sx$ ($x \in K$) is affine, i.e. it satisfies $s(\lambda x + (1-\lambda)y) = \lambda sx + (1-\lambda)sy$ for $s \in S$, $x, y \in K$ and $0 \leqslant \lambda \leqslant 1$.

3. Fixed point property of semigroup of non-expansive mappings

Suppose that S is a semitopological semigroup. We study in this section the relation between the existence of LIM for $WAP(S)$ and fixed point properties of S acting on certain subsets of a locally convex space. For the history and references regarding this topic one can see the survey article [25].

An action of a semitopological semigroup S on a Hausdorff space X is called *quasi-equicontinuous* if $\overline{S}^p$, the closure of S in the product space X^X, consists of only continuous mappings. Obviously, an equicontinuous action on a closed subset of a topological vector space is always quasi-equicontinuous (simply because if a net of equicontinuous functions converges pointwise to a function, then the limit function is also continuous). But a quasi-equicontinuous action on a convex compact subset of a topological vector space may not be equicontinuous. We will give a counterexample at the end of Section 4. The following properties of quasi-equicontinuity are obvious.

Lemma 3.1. *Let S be a semitopological semigroup that acts on a Hausdorff space X and the action is quasi-equicontinuous.*

(1) *If S_0 is a subsemigroup of S, then the action of S_0 on X is also quasi-equicontinuous*;
(2) *If in addition, X is compact, then for each compact S-invariant subspace X_0 of X, the action of S on X_0 is quasi-equicontinuous.*

Lemma 3.2. *Suppose that the action of S on a compact Hausdorff space X is separately continuous and quasi-equicontinuous. Then for each $x \in X$ and each $f \in C(X)$, we have $f_x \in WAP(S)$, where f_x is defined by*

$$f_x(s) = f(sx) \quad (s \in S).$$

Proof. Let βS be the spectrum of the C*-algebra $C(S)$. By [5, Theorem 4.2.3], it suffices to show that $\mathcal{RO}(f_x)$ is $\sigma(C(S), \beta S)$ pre-compact. Let $f \in C(X)$ be fixed and let $T : X \to C(S)$ be the mapping defined by $T(x) = f_x$ ($x \in X$). Then $T(Sx) = \mathcal{RO}(f_x)$. We shall show that T is continuous when $C(S)$ is equipped with $\sigma(C(S), \beta S)$ topology. This will imply our claim that $\mathcal{RO}(f_x)$ is $\sigma(C(S), \beta S)$ pre-compact since Sx is pre-compact.

For each $u \in \beta S$, there is a net $(s_\beta) \subset S$ such that δ_{s_β} converges to u in the weak* topology of $C(S)^*$, where δ_s denotes the point evaluation at s [5, Theorem 2.1.8]. Let $h \in C(X)$. Then

$$\langle h, s_\beta x\rangle = \langle h_x, \delta_{s_\beta}\rangle \to \langle h_x, u\rangle.$$

On the other hand, by assumption and by passing to a subnet if necessary, we may assume $s_\beta \to \hat{u}$ in X^X (since X^X is compact) and $\hat{u}$ is a continuous mapping $X \to X$. Thus, $s_\beta x$ is convergent in X to $\hat{u}(x)$ for every $x \in X$. In particular, $\langle h, s_\beta x\rangle \to \langle h, \hat{u}(x)\rangle$ for all $h \in C(X)$. Hence

$$\langle h_x, u\rangle = \big\langle h, \hat{u}(x)\big\rangle. \tag{1}$$

Now let (x_α) be a net in X and $x_\alpha \to x \in X$. Then

$$\begin{aligned} \big\langle u, T(x_\alpha)\big\rangle &= \langle u, f_{x_\alpha}\rangle = \big\langle \hat{u}(x_\alpha), f\big\rangle \\ &\to \big\langle \hat{u}(x), f\big\rangle = \langle u, f_x\rangle \\ &= \big\langle u, T(x)\big\rangle, \end{aligned}$$

by (1) and continuity of $\hat{u}$. So we have $T(x_\alpha) \to T(x)$ in $\sigma(C(S), \beta S)$ topology, as $x_\alpha \to x$. Consequently, T is continuous when $C(S)$ is equipped with $\sigma(C(S), \beta S)$ topology. Thus we have $f_x \in WAP(S)$ whenever $f \in C(X)$ and $x \in X$. □

Lemma 3.3. *Let S be a separable semitopological semigroup that acts on a weakly compact convex subset K of a locally convex space (E, Q) as weakly separately continuous and Q-non-expansive mappings. Suppose that F is a minimal non-empty weakly compact S-invariant subset of K satisfying $sF = F$ $(s \in S)$. Then F is Q-compact.*

Proof. It suffices to show that F is totally bounded in Q-topology.

Since F is non-empty minimal, we have $\overline{Sa}^{\mathrm{w}} = F$ $(a \in F)$. Let S_{c} be a countable dense subset of S. Then $\overline{Sa}^{\mathrm{w}} = \overline{S_{\mathrm{c}}a}^{\mathrm{w}}$ by the weak separate continuity. Moreover, $\overline{\mathrm{co}}^{\mathrm{w}}(Sa) = \overline{\mathrm{co}}^{\mathrm{w}}(S_{\mathrm{c}}a) = \overline{\mathrm{co}}(S_{\mathrm{c}}a)$ is separable in the Q-topology by Mazur's theorem and the fact that $S_{\mathrm{c}}a$ is countable. This shows that $\overline{\mathrm{co}}^{\mathrm{w}}(F)$ is closed and separable in the Q-topology.

Given a neighborhood N of 0 in (E, Q), there are finite seminorms $\{p_1, p_2, \ldots, p_n\} \subset Q$ and $\varepsilon > 0$ such that $U = \{x \in E\colon\ p_i(x) < \varepsilon, i = 1, 2, \ldots, n\}$ is a neighborhood of 0 contained in N. Take another Q-open symmetrical neighborhood V of 0 that satisfies $V + V \subset U$. Then there is a sequence $\{x_n\} \subset F$ such that $F \subset \bigcup_{n=1}^{\infty}\{x_n + V\}$ (due to the separability). From the Baire's category theorem, there is a weakly open neighborhood W of 0, an element $w \in F$ and an n such that $(w + W) \cap F \subset x_n + V$. This implies further that

$$(w + W) \cap F \subset w + (V + V) \subset w + U.$$

(Note $w \in x_n + V$ and hence $x_n \in w + V$.) Take a non-empty weakly open neighborhood W_1 of 0 such that $W_1 + W_1 \subset W$, and take finite semi-norms $\{\rho_1, \rho_2, \ldots, \rho_m\} \subset Q$ and a number $\delta > 0$ such that $H = \{x \in E\colon\ \rho_i(x) < \delta, i = 1, 2, \ldots, m\} \subset W_1$. Again due to the Q-separability, there is a sequence $\{y_n\} \subset F$ such that $F \subset \bigcup_{n=1}^{\infty}\{y_n + H\}$.

Since $\overline{Sa}^{\mathrm{w}} = F$, $w \in \overline{Sa}^{\mathrm{w}}$ for each $a \in F$. In particular, there is a sequence $\{s_n\} \subset S$ such that $s_1 y_1 \in w + W_1$, $s_2 s_1 y_2 \in w + W_1, \ldots, s_n s_{n-1} \cdots s_1 y_n \in w + W_1$ $(n = 1, 2, \ldots)$. From the Q-nonexpansiveness

$$\begin{aligned} s_n s_{n-1} \cdots s_1(y_n + H) \cap F &\subset s_n s_{n-1} \cdots s_1 y_n + H \\ &\subset (w + W_1 + W_1) \subset w + W. \end{aligned}$$

Therefore, $\{(s_n s_{n-1}\cdots s_1)^{-1}(w+W)\}_{n=1}^{\infty}$ is a weakly open cover of F. Since F is weakly compact, it has a finite subcover. Thus $F\subset\bigcup_{k=1}^{n}(s_k s_{k-1}\cdots s_1)^{-1}(w+W)$ for some integer n. From the assumption, $F=(s_n s_{n-1}\cdots s_1)F$. We then have

$$
\begin{aligned}
F &= \bigcup_{k=1}^{n}(s_n s_{n-1}\cdots s_{k+1})(w+W)\cap F \\
&\subset \bigcup_{k=1}^{n}(s_n s_{n-1}\cdots s_{k+1})(w+U)\cap F \\
&\subset \bigcup_{k=1}^{n}(s_n s_{n-1}\cdots s_{k+1}w+U),
\end{aligned}
$$

where the last inclusion is from the Q-nonexpansiveness. This shows that F is totally bounded Q-closed subset of E. Thus, F is Q-compact. □

Consider the following fixed point property.

(F) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E,Q) and the action is weakly separately continuous, weakly quasi-equicontinuous and Q-nonexpansive, then K contains a common fixed point for S.

We are now ready to prove our main theorem for this section.

Theorem 3.4. *Let S be a separable semitopological semigroup. Then $WAP(S)$ has a LIM if and only if S has the fixed point property* (F).

Proof. Assume that $WAP(S)$ has a LIM. Let X be a non-empty minimal weakly compact convex subset of K that is invariant under S and let $F\subset X$ be a non-empty minimal weakly compact subset of X that is invariant under S. Then, by Lemmas 3.1 and 3.2, $f_y\in WAP(S)$ for $f\in C(F)$ and $y\in F$. Here F is equipped with the weak topology inherited from E. Let Ψ be a LIM on $WAP(S)$ and define $\mu(f)=\Psi(f_y)$ $(f\in C(F))$. Then μ is a positive bounded linear functional on $C(F)$ satisfying $\mu(1)=1$ and $\mu({}_sf)=\mu(f)$ $(s\in S)$. From Riesz representation theorem, μ can be viewed as a regular probability measure on F and it satisfies $\mu(sA)=\mu(A)$ for each Borel set $A\subset F$ and $s\in S$. Let $\Gamma=\{A\subset F\colon A$ is weakly compact, $\mu(A)=1\}$. Then $sA\in\Gamma$ whenever $A\in\Gamma$. Let $F_0=\bigcap_{A\in\Gamma}A$. Then by finite intersection property F_0 is a non-empty weakly compact subset of F and $sF_0=\bigcap_{A\in\Gamma}sA\subset F_0$ for $s\in S$. We then have $F_0=F$ by the minimality of F. So Γ is a singleton. This implies $sF=F$ since $sF\in\Gamma$ for $s\in S$. From Lemma 3.3, F is Q-compact. We show that F contains only one point. The proof is in fact part of the proof of [22, Theorem 4.1] that comes from an idea of [10]. We include it here for the sake of completeness.

Suppose, to the contrary, that F has more than one point. Then there are $p\in Q$ and points $x_1,x_2\in F$ such that $r=p(x_1-x_2)=\sup\{p(x-y)\colon x,y\in F\}>0$. Let F_0 be the maximal subset of F containing x_1 and x_2 and satisfying $p(x-y)=r$ for all different $x,y\in F_0$. Then

F_0 is compact and hence must be finite. Let $F_0 = \{x_1, x_2, \ldots, x_n\}$. Let $\mu = \frac{1}{n}\sum_{i=1}^{n} x_i$. Then $\mu \in \mathrm{co}(F)$. Moreover, $p(\mu - x) \leqslant r$ for all $x \in F$. In fact,

$$r_0 = \sup\{p(\mu - x)\colon\ x \in F\} < r$$

because otherwise there would be a sequence $(y_i) \subset F$ such that $p(\mu - y_i) \to r$. By passing to a subsequence we may assume $y_i \to y_0 \in F$. Then $p(\mu - y_0) = r$. This implies that $p(x_i - y_0) = r$ for all $i = 1, 2, \ldots, n$, which contradicts the maximality of F_0. So $r_0 < r$. Let

$$M = \{x \in X\colon\ p(x - y) \leqslant r_0 \text{ for all } y \in F\}.$$

Then $\mu \in M$. M is a nonempty Q-closed convex (hence is also weakly closed) proper subset of X. If $x \in M$, then $p(x - y) \leqslant r_0$ $(y \in F)$. From the Q-nonexpansiveness, $p(sx - sy) \leqslant r_0$ $(y \in F)$. This leads to $p(sx - y) \leqslant r_0$ $(s \in S,\ y \in F)$ since $F = sF$. As a consequence, $sx \in M$ $(s \in S,\ x \in M)$ and hence M is S-invariant. This is a contradiction to the minimality of X. Thus, F contains exactly one point, which, of course, must be a common fixed point for S.

To prove the converse, we first note that weak continuity implies weak quasi-equicontinuity if the action on K is affine and τ-equicontinuous, where τ is the locally convex topology on E induced by Q. To see the latter, we assume that there is a net $(s_\alpha) \subset S$ satisfying $s_\alpha x \xrightarrow{\text{wk}} T(x)$ for each $x \in K$. We show that T is weak–weak continuous. Otherwise we would have a net $(x_\beta) \subset K$ such that $x_\beta \xrightarrow{\text{wk}} x \in K$ but $T(x_\beta) \overset{\text{wk}}{\nrightarrow} T(x)$. Then there would be $f \in E^*$, a $\varepsilon > 0$ and a subnet of (x_β) (still denoted by (x_β)) such that $\mathrm{Re}(\langle f, T(x_\beta) - T(x)\rangle) > \varepsilon$ for all β while $x_\beta \xrightarrow{\text{wk}} x$. By Mazur's theorem, there would be a net $(x_\lambda) \subset \mathrm{co}(x_\beta)$ such that $x_\lambda \xrightarrow{\tau} x$. We certainly still have $\mathrm{Re}(\langle f, T(x_\lambda) - T(x)\rangle) > \varepsilon$ for all λ since the S action is affine and

$$\langle f, T(x_\lambda) - T(x)\rangle = \lim_\alpha \langle f, s_\alpha x_\lambda - s_\alpha x\rangle.$$

But from the τ-equicontinuity (actually, only τ-weak equicontinuity is needed here), there is a λ_0 such that $|\langle f, s_\alpha x_\lambda - s_\alpha x\rangle| < \varepsilon$ for all α and $\lambda > \lambda_0$. Therefore, $\|\langle f, T(x_\lambda) - T(x)\rangle\| \leqslant \varepsilon$ $(\lambda > \lambda_0)$, which is a contradiction.

Now let $E = WAP(S)^*$ with the topology determined by the family of continuous semi-norms $Q = \{p_f\colon\ f \in WAP(S)\}$, where

$$p_f(\phi) = \sup\{|\phi(\ell_s f)|, |\phi(f)|\colon\ s \in S\} \quad (\phi \in E). \tag{2}$$

Then by Mackey–Aren's theorem (see [39]) the weak topology of (E, Q) and the weak*-topology $\sigma(WAP(S)^*, WAP(S))$ coincide. Let $K =$ all means on $WAP(S)$. Then K is a weakly compact convex subset of (E, Q). Consider the S action on E defined by $s \mapsto \ell_s^*$ $(s \in S)$, where ℓ_s^* denotes the dual operator of the translation operator $\ell_s : WAP(S) \to WAP(S)$. One can verify that this action is separately continuous and it gives a representation of S as weakly separately continuous, Q-nonexpansive mappings on K. On the other hand, the action is also affine on K. Hence, $\overline{S}^p \subset C(K^{\mathrm{w}}, K^{\mathrm{w}})$. Apply (F) for this E and K. We then are ensured a common fixed point in K for S. This fixed point is certainly a left invariant mean on $WAP(S)$. The proof is complete. □

Remark 3.5. Consider the following fixed point property for a semitopological semigroup S:

(E) Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) as Q-nonexpansive self-mappings and, if in addition, the action is separately continuous and equicontinuous when K is equipped with the weak topology of (E, Q), then K contains a common fixed point for S.

Clearly we have

$$\text{(G)} \quad \Rightarrow \quad \text{(F)} \quad \Rightarrow \quad \text{(E)} \quad \Rightarrow \quad \text{(D)}.$$

Open problem. Can any of the above implications be reversed?

When S is discrete and left reversible, then (G) holds as shown by Hsu [19].

Using the method of Theorem 3.4 we have another characterization for $AP(S)$ on a separable semigroup S to have a LIM.

Theorem 3.6. *Let S be a separable semitopological semigroup. Then $AP(S)$ has a LIM if and only if the fixed point property* (E) *holds.*

Another result that can be derived by using Lemma 3.2 and by combining the arguments of Theorem 3.4 and [22, Theorem 3.5] is the following.

Theorem 3.7. *Let S be separable and n be a positive integer. Then $WAP(S)$ has a LIM of the form $\frac{1}{n}\sum_{i=1}^{n}\phi_i$, where each ϕ_i is a multiplicative mean on $WAP(S)$, if and only if*

(P_n) *Whenever S is a separately continuous and quasi-equicontinuous action on a compact Hausdorff space X, then there exists a nonempty finite subset $F \subseteq X$, $|F| \leqslant n$, $|F|$ divides n such that $sF = F$ for all $s \in S$.*

For $n = 1$, in particular we have:

Theorem 3.8. *$WAP(S)$ has a multiplicative LIM if and only if whenever S is a separately continuous and quasi-equicontinuous action on a compact Hausdorff space X, then X has a common fixed point for S.*

Let (E, Q) be a separable locally convex space. A subset K of E is said to have Q-normal structure (for Banach space case see [2,6]) if, for each Q-bounded subset H of K that contains more than one point, there is $x_0 \in \operatorname{co} H$ and $p \in Q$ such that $\sup\{p(x - x_0)\colon x \in H\} < \sup\{p(x - y)\colon x, y \in H\}$. Here by Q-boundedness of H we mean for each $p \in Q$ there is $d > 0$ such that $p(x) \leqslant d$ for all $x \in H$. Any Q-compact subset has Q-normal structure. In a uniformly convex space (e.g. any L^p, $p > 1$, space) a bounded convex set always has normal structure.

Theorem 3.9. *Let S be a semitopological semigroup. Then $AP(S)$ has LIM if and only if S has the following fixed point property.*

(E′) *Whenever S acts on a weakly compact convex subset K of a separated locally convex space (E, Q) as Q-nonexpansive mappings, if K has Q-normal structure and the S-action is* *separately continuous and equicontinuous when K is equipped with the weak topology of (E, Q), then K contains a common fixed point for S.*

In particular, fixed point properties (D) *and* (E′) *are equivalent.*

Proof. Suppose that $AP(S)$ has a LIM Φ. Let X be a minimal non-empty weakly closed convex subset of K invariant under S action, and let F be a minimal non-empty weakly closed subset of X invariant under S action. From [22, Lemma 3.1], $f_y \in AP(S)$ for each $f \in C(F)$ and $y \in F$. So μ defined by $\mu(f) = \Phi(f_y)$ is a mean on $C(F)$. Following the argument of Theorem 3.4, one sees that $sF = F$ ($s \in S$). On the other hand, F is Q-bounded since it is weakly compact. If F contains more than one point, by the normal structure of K, there is $x_0 \in \operatorname{co} F$ and $p \in Q$ such that

$$r_0 = \sup\bigl\{p(x - x_0)\colon\ x \in F\bigr\} < r = \sup\bigl\{p(x - y)\colon\ x, y \in F\bigr\}.$$

Then the same argument as in the proof of Theorem 3.4 leads to a contradiction, showing that F is a singleton. The point in F is a common fixed point for S. For the converse, suppose that (E′) holds. Let $E = AP(S)^*$ with the topology determined by $Q = \{p_f\colon\ f \in AP(S)\}$, where p_f is defined as in (3.2). Let K be the set of all means on $AP(S)$. Then K is Q-compact since Q-topology coincides with the weak*-topology on K and K is weak*-compact. Thus K has the Q-normal structure [18, Lemma 2]. Moreover, the action $s \to \ell_s^*$ of S on K is certainly separately continuous and equicontinuous when K is equipped with the weak topology of (E, Q). Therefore K has a common fixed point for S, which is a LIM on $AP(S)$. □

Note that the condition of (E) is weaker than that of (E′), since we do not require Q-normal structure. Theorems 3.6 and 3.9 show that they are equivalent when the semigroup S is separable, i.e. when S contains a countable dense subset. A known result for this kind of characterization is [22, Theorem 3.2] which asserts that $AP(S)$ has a LIM if and only if the fixed point property (E) with Q-nonexpensiveness replaced by affiness of the action of S on K holds.

4. Amenability of bicyclic semigroups

In this section, we shall study the class of bicyclic semigroups and partially bicyclic semigroups and use this to give an example of a semigroup which is not left reversible but has fixed point property (F). We also give an example of a semigroup S such that $AP(S)$ has a LIM but $WAP(S)$ does not have a LIM. In particular, we answer Problem 27 in [24] and Problem 1 in [23].

The bicyclic semigroup is the semigroup generated by a unit e and two more elements p and q subject to the relation $pq = e$. We denote it by $S_1 = \langle e, p, q \mid pq = e\rangle$. The semigroup generated by a unit e and three more elements a, b and c subject to the relations $ab = ac = e$ is denoted by $S_2 = \langle e, a, b, c \mid ab = e, ac = e\rangle$; and the semigroup generated by a unit e and four more elements a, b, c, d subject to the relations $ac = bd = e$ is denoted by $S_{1,1} = \langle e, a, b, c, d \mid ac = e, bd = e\rangle$. S_2 and $S_{1,1}$ will be called partially bicyclic semigroups. Duncan and Namioka showed in [13] that S_1 is an amenable semigroup by revealing the maximal group homomorphic image of S_1. Here we can prove the same result directly by constructing a left and a right invariant mean on $\ell^\infty(S_1)$.

Proposition 4.1. *The bicyclic semigroup S_1 is amenable.*

Proof. For any $\varepsilon > 0$ and a finite set

$$F = \{q^{m_i} p^{n_i}\colon\ m_i \geqslant 0,\ n_i \geqslant 0,\ i = 1, 2, \dots, l\} \subset S_1,$$

let

$$m = \max\{n_i, m_i - n_i\colon\ i = 1, 2, \dots, l\} \quad \text{and} \quad k = 2mt,$$

where $t > 1/\varepsilon$ is an integer. Then setting

$$A = \{q^m, q^{m+1}, \dots, q^{m+k}\} \subset S_1$$

we have that for any $s = q^{m_i} p^{n_i} \in F$,

$$sA = \{q^{m+m_i-n_i}, q^{m+m_i-n_i+1}, \dots, q^{m+m_i-n_i+k}\}.$$

So $|A| = k + 1$, $|A \sim sA| \leqslant m$ and $|sA \sim A| \leqslant m$. Define $\Phi_{F,\varepsilon} = \frac{1}{|A|}\chi_A$, where for a subset E, χ_E denotes the characteristic function of E. Then

$$\begin{aligned} \|s * \Phi_{F,\varepsilon} - \Phi_{F,\varepsilon}\|_1 &= \frac{1}{|A|}\|\chi_{sA} - \chi_A\|_1 \\ &= \frac{1}{|A|}\big(|A \sim sA| + |sA \sim A|\big) \leqslant \frac{2m}{k+1} < \varepsilon \end{aligned}$$

for $s \in F$. Let $\Lambda = \{(F, \varepsilon)\colon\ F \subset S_1 \text{ is finite}, \varepsilon > 0\}$ with the usual partial order

$$\alpha_1 = (F_1, \varepsilon_1) \geqslant \alpha_2 = (F_2, \varepsilon_2) \quad \text{iff} \quad F_1 \supseteq F_2 \quad \text{and} \quad \varepsilon_1 \leqslant \varepsilon_2.$$

Then $(\Phi_\alpha)_{\alpha\in\Lambda} \subset \ell^1(S_1)$ satisfies $\|\Phi_\alpha\|_1 = 1$ and

$$\|s * \Phi_\alpha - \Phi_\alpha\|_1 \xrightarrow{\alpha} 0 \quad (s \in S_1).$$

This shows that every weak* cluster point of $(\Phi_\alpha)_{\alpha\in\Lambda}$ in $(\ell^1(S_1))^{**}$ gives a left invariant mean on $\ell^\infty(S_1)$. Similarly there is a right invariant mean on $\ell^\infty(S_1)$ (to see this one needs only to replace q in the set A with p and interchange m_i and n_i in the definition of the integer m). Therefore S_1 is both left and right amenable and hence is amenable. □

Remark 4.2. We note that S_1 is neither left nor right cancellative hence not embeddable into a group.

Proposition 4.3. *The partially bicyclic semigroups S_2 and $S_{1,1}$ are not left amenable.*

Proof. This is simply because both S_2 and $S_{1,1}$ are not left reversible. For instance, in S_2 we have $bS_2 \cap cS_2 = \emptyset$; and in $S_{1,1}$ we have $bS_{1,1} \cap dS_{1,1} = \emptyset$. □

Because of the symmetry in the structure of $S_{1,1}$, we see that $S_{1,1}$ is also not right amenable. However, the situation for S_2 is different. We have the following.

Proposition 4.4. *The partially bicyclic semigroup S_2 is right amenable.*

Proof. The argument is similar to that for S_1. Let F be a finite set of S_2. We can write

$$F = \{f_1 a^{m_1}, f_2 a^{m_2}, \dots, f_n a^{m_n}\},$$

where each $f_i \in \langle e, b, c\rangle$, and $m_i \geqslant 0$, $i = 1, 2, \dots, n$. Denote the length of any element $f \in \langle e, b, c\rangle$ by $l(f)$. Given $\varepsilon > 0$, take

$$m = \max\{l(f_i), m_i - l(f_i) \mid i = 1, 2, \dots, n\} \quad \text{and} \quad k = 2mt,$$

where $t > 1/\varepsilon$ is an integer. Define

$$A = \{a^m, a^{m+1}, \dots, a^{m+k}\}.$$

Then for $f_i a^{m_i} \in F$, we have

$$A \cdot f_i a^{m_i} = \{a^{m+m_i-l(f_i)}, a^{m+m_i-l(f_i)+1}, \dots, a^{m+m_i-l(f_i)+k}\}.$$

Thus $|A \varDelta (A \cdot f_i a^{m_i})| \leqslant 2m$. Define $\varPhi_{F,\varepsilon} = \frac{1}{|A|}\chi_A$. We then have

$$\|\varPhi_{F,\varepsilon} \cdot s - \varPhi_{F,\varepsilon}\|_1 = \frac{1}{|A|}|A \varDelta A \cdot s| \leqslant \frac{2m}{k+1} \leqslant \varepsilon$$

for all $s \in F$. This implies that there exists a right invariant mean on $\ell^\infty(S_2)$. □

For a discrete semigroup S it is known that if S is left reversible, then $WAP(S)$ has a LIM, which in turn implies that $AP(S)$ has a LIM. Whether or not the converse is true is an open question [24, Problem 27]. T. Mitchell [38] proved in 1984 that both $AP(S_2)$ and $AP(S_{1,1})$ have an invariant mean by using the Swelling lemma (see [17, A-1.20]). Note that both S_2 and $S_{1,1}$ are not left reversible. So they provide examples of a discrete semigroup S that is not left reversible but $AP(S)$ has a left invariant mean. Since the proof of Mitchell's has never been published, here we include a proof for completion. Recall that a *topological semigroup* is a semigroup with a Hausdorff topology such that the multiplication is jointly continuous. We first state the Swelling lemma as following.

Lemma 4.5. *Let S be a compact topological semigroup. If $X \subset S$ and $s \in S$ are such that $X \subset sX$, then $X \subset sX \subset \overline{X}$.*

Proposition 4.6. *Let $S_2 = \langle e, a, b, c \mid ab = e, ac = e\rangle$ and $S_{1,1} = \langle e, a, b, c, d \mid ac = e, bd = e\rangle$. Then both $AP(S_2)$ and $AP(S_{1,1})$ have an invariant mean. In particular, both S_2 and $S_{1,1}$ have fixed point property* (D).

Proof. Denote either of S_2 and $S_{1,1}$ simply by S. To show that $AP(S)$ has an invariant mean, it suffices to show that S^{a}, the almost periodic compactification of S, is a compact group. Since S^{a} is a compact topological semigroup, we need only to show that S^{a} is a group. To this end we prove that for every $s \in S^{\mathrm{a}}$, $S^{\mathrm{a}} = sS^{\mathrm{a}}$. A similar argument will give that $S^{\mathrm{a}} = S^{\mathrm{a}}s$ ($s \in S^{\mathrm{a}}$). Then we can conclude that $e \in sS^{\mathrm{a}} \cap S^{\mathrm{a}}s$ ($s \in S^{\mathrm{a}}$), showing that s is both left and right invertible and hence is invertible. From the definition of S_2 and $S_{1,1}$ it is easy to see that

$$aS_2 = S_2, \quad \text{and} \quad bS_{1,1} = aS_{1,1} = S_{1,1}.$$

By continuity, we have

$$aS_2^{\mathrm{a}} = S_2^{\mathrm{a}}, \quad \text{and} \quad bS_{1,1}^{\mathrm{a}} = aS_{1,1}^{\mathrm{a}} = S_{1,1}^{\mathrm{a}}.$$

Now for $S = S_2$ let $X = bS^{\mathrm{a}}$ and $s = a$. Then $sX = S^{\mathrm{a}}$. By Swelling lemma $S^{\mathrm{a}} = sX \subset \overline{X} = X = bS^{\mathrm{a}}$. Similarly, $cS^{\mathrm{a}} = S^{\mathrm{a}}$. For $S = S_{1,1}$, applying Swelling lemma to, respectively, the pair $X = cS^{\mathrm{a}}$, $s = a$ and the pair $X = dS^{\mathrm{a}}$, $s = b$, we have that $S^{\mathrm{a}} = cS^{\mathrm{a}} = dS^{\mathrm{a}}$. We therefore have shown that $sS^{\mathrm{a}} = S^{\mathrm{a}}$ for each generator element of S, when $S = S_2$ or $S = S_{1,1}$. This certainly implies that $sS^{\mathrm{a}} = S^{\mathrm{a}}$ for all $s \in S$. The latter, in turn, implies further that $sS^{\mathrm{a}} = S^{\mathrm{a}}$ for all $s \in S^{\mathrm{a}}$. □

We now aim to show that $WAP(S_2)$ has a LIM while $WAP(S_{1,1})$ has no LIM. This will answer the open question stated before Lemma 4.5. We deal with $WAP(S_2)$ first.

Let S be a semigroup. We denote by S^{w} the weakly almost periodic compactification of S. It is known that S^{w} is a compact (universal) semitopologic semigroup [5] containing S as a dense subgroup. Let A be the set of all limit points of the subsemigroup $\langle a\rangle$ in S_2^{w}, i.e.,

$$A = \bigcap_{n=1}^{\infty} \overline{A_n}, \quad A_n = \{a^n, a^{n+1}, a^{n+2}, \dots\}.$$

Then $A \neq \emptyset$ and is a compact abelian subsemigroup of S_2^{w}.

Lemma 4.7. *The subsemigroup A is a right ideal of S_2^{w}.*

Proof. Suppose that $s \in S_2$ and $u \in A$. Let $s = ta^m$ and $u = \lim a^{m_i}$, where $t \in \langle e, b, c\rangle$, $m \geqslant 0$, is an integer and $(a^{m_i}) \subset \langle a\rangle$. Then by separate continuity,

$$us = \lim_i a^{m_i} t a^m = \lim_i a^{m_i + m - l(t)} \in A.$$

Therefore $AS_2 \subset A$. Thus $AS_2^{\mathrm{w}} \subset A$ since S_2 is dense in S_2^{w} and A is closed. □

Lemma 4.8. *The subsemigroup A has a unique minimal idempotent e_A which is also a minimal idempotent of S_2^{w}.*

Proof. Since A is a compact abelian semitopological semigroup, it has a unique minimal idempotent e_A. From Lemma 4.7 A is a closed right ideal of S_2^{w}. So A contains a minimal idempotent of S_2^{w}, which, of course, is also a minimal idempotent of A. Therefore e_A is the only minimal idempotent of S_2^{w} contained in A. □

Lemma 4.9. *Let e_A be the idempotent obtained in Lemma* 4.8. *Then $be_A = e_A b$ and $ce_A = e_A c$.*

Proof. If $be_A \neq e_A b$, then there would exist $f \in WAP(S_2)$ such that $f(be_A) = 0$ and $f(e_A b) = 1$. Let $e_A = \lim_i a^{m_i}$ in the weak* topology of $WAP(S_2)^*$. We have

$$\lim_i f\left(ba^{m_i}\right) = 0, \qquad \lim_i f\left(a^{m_i - 1}\right) = 1.$$

So there is an increasing subsequence of the net (m_i), denoted also by (m_i), for which the above two limits hold. Consider

$$f\left(a^{m_i-1}b^n a^n\right) = \ell_{a^{m_i-1}}\left(r_{b^n a^n}(f)\right)(1) \quad (n \in \mathbb{N}).$$

By the Eberlein–Smulian theorem [5, Theorem A.5, (ii) $\Leftrightarrow$ (iii)], there is a subsequence of $\mathbb{N}$, say (n_j), such that $r_{b^{n_j}a^{n_j}}(f)$ converges to some $h \in WAP(S_2)$ weakly. So for each m_i,

$$\ell_{a^{m_i-1}}\left(r_{b^{n_j}a^{n_j}}(f)\right) \xrightarrow{j} \ell_{a^{m_i-1}}(h)$$

weakly. In particular,

$$\lim_j \ell_{a^{m_i-1}}\left(r_{b^{n_j}a^{n_j}}(f)\right)(1) = \ell_{a^{m_i-1}}(h)(1) = h\left(a^{m_i-1}\right).$$

Passing to a subsequence of (m_i) if necessary, we can assume that $\lim_i h(a^{m_i-1})$ exists. This shows that the iterated sequence limit

$$I = \lim_i \lim_j f\left(a^{m_i-1}b^{n_j}a^{n_j}\right)$$

exists. We now use different ways to calculate the value I.

Way 1. Let $s_i = a^{m_i-1}$ and $t_j = b^{n_j}a^{n_j}$. Then

$$s_i t_j = \begin{cases} a^{m_i-1}, & \text{if } m_i > n_j, \\ a^{m_i-1}b^{n_j}a^{n_j}, & \text{if } m_i \leqslant n_j. \end{cases}$$

From the double limit characterization of weak almost periodicity [5, Theorem 4.2.3], we have

$$I = \lim_i \lim_j f(s_i t_j) = \lim_j \lim_i f(s_i t_j) = \lim_j \lim_i f\left(a^{m_i-1}\right) = 1.$$

Way 2. Let $s'_i = ba^{m_i}$ and $t'_j = b^{n_j}a^{n_j}$. Then

$$s'_i t'_j = \begin{cases} ba^{m_i}, & \text{if } m_i > n_j, \\ b^{n_j - m_i + 1}a^{n_j} = a^{m_i-1}b^{n_j}a^{n_j}, & \text{if } m_i \leqslant n_j. \end{cases}$$

Therefore

$$I = \lim_i \lim_j f\left(s'_i t'_j\right) = \lim_j \lim_i f\left(s'_i t'_j\right) = \lim_j \lim_i f\left(ba^{m_i}\right) = 0.$$

This contradicts the conclusion we got from Way 1.

So we have shown that $be_A = e_A b$. A similar argument shows that the other equality $ce_A = e_A c$ holds too. $\square$

Corollary 4.10. *For every $s \in S_2^{\mathrm{w}}$, $se_A = e_A s$.*

Proof. This is a straightforward consequence of Lemma 4.9. $\square$

Theorem 4.11. *The weakly almost periodic compactification S_2^{w} of S_2 has a unique minimal idempotent. The minimal ideal $K(S_2^{\mathrm{w}})$ of S_2^{w} is a group and $WAP(S_2)$ has a LIM. In particular, S_2 has fixed point property* (F) *and S_2 is not left reversible.*

Proof. From Proposition 4.4 $WAP(S_2)$ has a right invariant mean. So S_2^{w} has a unique minimal left ideal. From [5, Corollary 1.2.17] $E(K(S_2^{\mathrm{w}}))$, the set of minimal idempotents of S_2^{w}, is a left zero semigroup, i.e., $e_1 e_2 = e_1$ for all $e_1, e_2 \in E(K(S_2^{\mathrm{w}}))$. By Lemma 4.8 $e_A \in E(K(S_2^{\mathrm{w}}))$. Then for any $e \in E(K(S_2^{\mathrm{w}}))$, by Corollary 4.10 we have

$$e = ee_A = e_A e = e_A.$$

Therefore e_A is the only minimal idempotent of S_2^{w}. This in turn implies that S_2^{w} has a unique minimal right ideal. From [5, Corollary 1.5.2.(ii)] the minimal ideal $K(S_2^{\mathrm{w}})$ of S_2^{w} is a compact topological group. Then the integral over $K(S_2^{\mathrm{w}})$ with respect to a Haar measure on $K(S_2^{\mathrm{w}})$ gives a LIM for $WAP(S_2)$ (in fact, it gives an invariant mean on $WAP(S_2)$). $\square$

The remainder of the section is devoted to study the semigroup $S_{1,1}$. Let

$$A = dbcS_{1,1} = \{\text{all words in } S_{1,1} \text{ starting with } dbc\}.$$

In the sequel, when we represent an element $s \in S_{1,1}$ as a word, we always assume the representation is irreducible.

Lemma 4.12. *The characteristic function χ_A of A is weakly almost periodic.*

Proof. From the double limit criterion, it suffices to show either of the following is true:

$$\lim_m \lim_n \chi_A(s_m t_n) = \lim_n \lim_m \chi_A(s_m t_n) = 0, \tag{3}$$

$$\lim_m \lim_n \chi_A(s_m t_n) = \lim_n \lim_m \chi_A(s_m t_n) = 1, \tag{4}$$

where $(s_m), (t_n) \subset S_{1,1}$ are sequences such that the two iterated limits involved exist.

Case 1. Suppose that there are infinitely many s_m belonging to A. Then, clearly, (4) holds. So in the remainder cases we can assume $s_m \notin A$ for all m.

Case 2. Suppose that there are infinitely many s_m containing c or infinitely many s_m containing d not as the first letter. Then this c or letter(s) before this d will remain in the word representation of the product $s_m t_n$, preventing it to begin with dbc. Therefore (3) holds for this case.

We then have two cases left: $s_m \in d\langle e, a, b\rangle$ for all m or $s_m \in \langle e, a, b\rangle$ for all m.

Case 3. Suppose $s_m \in d\langle e, a, b\rangle$ for all m. If there are infinitely many n such that t_n does not contain c, then obviously (3) holds. Assume t_n contains c for all n. Then we can write

$$t_n = \tau_n c \tau_n', \quad \tau_n \in \langle e, a, b, d\rangle.$$

If $(l(\tau_n))$ is unbounded, where $l(s)$ denotes the length of the reduced word representation of s, then (3) holds no matter $(l(s_m))$ is bounded or not.

Assume $(l(\tau_n))$ is bounded. Then (3) holds if $(l(s_m))$ is unbounded. Now suppose both $(l(s_m))$ and $(l(\tau_n))$ are bounded. Then there is $u \in \langle e, a, b\rangle$ such that $s_m = du$ for infinitely many m. This certainly implies that

$$\lim_m \lim_n \chi_A(s_m t_n) = \lim_n \lim_m \chi_A(s_m t_n) = \lim_m \lim_n \chi_A(d u t_n).$$

Note that the two iterated limits are assumed to exist. So either (3) or (4) holds for this case. This completes the proof for Case 3.

Case 4. Suppose $s_m \in \langle e, a, b\rangle$ for all m. If there are infinitely many t_n that does not contain the segment dbc, then (3) holds. We then can assume all t_n contain the segment dbc. So we can write $t_n = \tau_n dbc \tau_n'$, where τ_n does not contain dbc.

If $(l(\tau_n))$ is unbounded, then obviously (3) holds regardless $(l(s_m))$ being bounded or not. If $(l(\tau_n))$ is bounded, then τ_n repeatedly take some word v infinite times, i.e. $t_n = v dbc \tau_n'$ for infinitely many n. In this case (3) holds if $v \notin \langle e, d, c\rangle$. Suppose that $v \in \langle e, d, c\rangle$. If there are infinitely many s_m such that $s_m v \neq e$, then (3) holds. Otherwise, we may assume $s_m v = e$ for all m. Then (4) holds. This shows our claim for Case 4 and hence completes the proof. □

Theorem 4.13. *$WAP(S_{1,1})$ has no LIM. In particular, $S_{1,1}$ has fixed point property* (D) *but not fixed point property* (F).

Proof. Let $K(\chi_A)$ be the closure of $\mathrm{co}(\mathcal{RO}(\chi_A))$ under the topology of pointwise convergence. Then for each $f \in K(\chi_A)$ we have $f(dbc) = 1$ and $f(c) = 0$. So $K(\chi_A)$ contains no constant function. Since $\chi_A \in WAP(S_{1,1})$ from the preceding lemma, the result follows from [5, Theorem 2.3.11]. □

With Theorems 4.11 and 4.13 we can complete the diagram in Section 2 as follows:

$$\begin{array}{c} S \text{ left amenable} \\ \Downarrow \;\; \not\Uparrow \\ S \text{ left reversible} \\ \Downarrow \;\; \not\Uparrow \\ WAP(S) \text{ has LIM} \\ \Downarrow \;\; \not\Uparrow \\ AP(S) \text{ has LIM} \end{array}$$

We now can give an example of a quasi-equicontinuous but not equicontinuous action of a semigroup S on a compact convex subset of a separated locally convex space. We have shown that

$AP(S_{1,1})$ has LIM but $WAP(S_{1,1})$ has no LIM. In other words, $S_{1,1}$ has fixed point property (D) but has no fixed point property (F).

Let K be the set of all means on $WAP(S_{1,1})$. K is a compact convex subset of $E = WAP(S_{1,1})^*$ under the weak topology of the locally convex topological space (E, Q), where Q is the family of semi-norms as defined in the proof of Theorem 3.4.

Example 4.14. Let K be as above and be equipped with the weak topology of (E, Q). The action of $S_{1,1}$ on K defined by $s \mapsto \ell_s^*$ $(s \in S_{1,1})$ is quasi-equicontinuous but is not equicontinuous.

Proof. The quasi-equicontinuity of the action has been proved (for general case) in the proof of Theorem 3.4. On the other hand, the action is clearly affine. If it were equicontinuous, K would have a common fixed point for $S_{1,1}$ due to [22, Theorem 3.2], which would be a LIM on $WAP(S_{1,1})$, a contradiction. □

5. Jointly continuous actions

In this section, we shall consider fixed point properties (F^*) and (G^*), where separate continuity in (F) and (G) are replaced by joint continuity, respectively. Clearly, the fixed point property (G^*) implies the fixed point property (F^*).

Theorem 5.1. *Let* (F^*) *denote the fixed point property* (F) *with weak separate continuity replaced by weak joint continuity. Suppose that S is a separable semitopological semigroup. Then $WAP(S) \cap LUC(S)$ has a LIM if and only if the fixed point property* (F^*) *holds.*

Proof. Let K be the weakly compact convex set described in the property (F^*). If the action of S on K is weakly joint continuous and weakly quasi-equicontinuous, then $f_y \in WAP(S) \cap LUC(S)$ for $f \in C(F)$ and $y \in F$ from Lemma 3.2, joint continuity and the weak pre-compactness of Sy, where F is a non-empty minimal weakly compact S-invariant subset of K and is equipped with the weak topology of (E, Q). Hence the argument for the necessity of Theorem 3.4 is valid with $WAP(S)$ being replaced by $WAP(S) \cap LUC(S)$. This shows that if $WAP(S) \cap LUC(S)$ has a LIM, then the fixed point property (F^*) holds. For the converse, we note that for $E = (WAP(S) \cap LUC(S))^*$ and $K =$ the set of all means on $WAP(S) \cap LUC(S)$, the action $s \mapsto \ell_s^*$ on K is weakly jointly continuous. So the sufficiency part of the proof of Theorem 3.4 still holds if $WAP(S)$ is replaced by $WAP(S) \cap LUC(S)$. □

We call a semitopological semigroup S *strongly left reversible* if there is a family of countable subsemigroups $\{S_\alpha\colon \alpha \in I\}$ such that:

(1) $S = \bigcup_{\alpha \in I} S_\alpha$,
(2) $\overline{aS_\alpha} \cap \overline{bS_\alpha} \neq \emptyset$ for each $\alpha \in I$ and $a, b \in S_\alpha$,
(3) for each pair $\alpha_1, \alpha_2 \in I$, there is $\alpha_3 \in I$ such that $S_{\alpha_1} \cup S_{\alpha_2} \subset S_{\alpha_3}$.

Obviously, if S is strongly left reversible then it is left reversible, and a left reversible separable semigroup is strongly left reversible. Hsu [19] showed that a discrete left reversible semigroup is always strongly left reversible. Using his idea we have the following.

Lemma 5.2. *A metrizable left reversible semitopological semigroup is strongly left reversible.*

Proof. Let S be such a semigroup. For a subset $B \subset S$, we denote by $\langle B \rangle$ the semigroup generated by B. It suffices to show that for each finite set $B \subset S$, there exists a countable subsemigroup S_B such that $B \subset S_B$ and $\overline{aS_B} \cap \overline{bS_B} \neq \emptyset$ $(a, b \in S_B)$.

Given $B \subset S$ finite, let $S_1 = \langle B \rangle$. Then S_1 is countable. Denote by J the collection of all finite subsets of S_1. J is countable. For $A = \{a_1, a_2, \ldots, a_n\} \in J$, there is $c \in \overline{a_1 S} \cap \overline{a_2 S} \cap \cdots \cap \overline{a_n S}$. So there are sequences $\{b_{ij}\}_{j=1}^{\infty}$ $(i = 1, 2, \ldots, n)$ such that

$$c = \lim_{j \to \infty} a_i b_{i,j} \quad (i = 1, 2, \ldots, n),$$

since S is metrizable. Let

$$V_A = \langle A, b_{ij}\colon\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots \rangle.$$

Then V_A is countable and $c \in \overline{V_A}$. Let $S_2 = \langle \bigcup_{A \in J} V_A \rangle$. Then S_2 is countable, $S_1 \subset S_2$ and $\overline{aS_2} \cap \overline{bS_2} \neq \emptyset$ for $a, b \in S_1$. Repeat the above procedure. We then have an increasing sequence of countable subsemigroups

$$S_1 \subset S_2 \subset S_3 \subset \cdots \subset S_m \subset S_{m+1} \subset \cdots$$

satisfying $\overline{aS_{m+1}} \cap \overline{bS_{m+1}} \neq \emptyset$ for $a, b \in S_m$. Now we take $S_B = \bigcup_{m=1}^{\infty} S_m$. Then $B \subset S_B$, S_B is countable, and $\overline{aS_B} \cap \overline{bS_B} \neq \emptyset$ $(a, b \in S_B)$. $\square$

Lemma 5.3. *Suppose that S is a semitopological semigroup that acts on a compact Hausdorff space X and the action $S \times X \to X$ is jointly continuous. If S contains a dense subset D such that $\overline{aS} \cap \overline{bS} \neq \emptyset$ for $a, b \in D$, then any minimal S-invariant non-empty compact subset K of X* (*clearly such K does exist from Zorn's lemma*) *satisfies*:

(1) $\overline{Sx} = K$ *for all* $x \in K$,
(2) $sK = K$ *for all* $s \in S$.

Proof. Let $K \neq \emptyset$ be a minimal S-invariant compact subset of X. Then for each $x \in K$, $\overline{Sx}$ is a compact S-invariant subset of K. So $\overline{Sx} = K$ by minimality of K, i.e. (1) holds. To show (2), we first note that, given $x \in K$, the collection $\{\overline{sSx}\colon\ s \in D\}$ has finite intersection property. Thus,

$$Y = \bigcap \bigl\{\overline{sSx}\colon\ s \in D\bigr\} \subset \bigcap_{s \in D} sK$$

is a non-empty compact subset of K. We show $aY \subset Y$ for all $a \in D$. To this end we need to show that, for each fixed $a \in D$ and $y \in Y$, $ay \in \overline{bSx}$ for all $b \in D$. In fact, given such a and y, for each $b \in D$ there exists $c \in \overline{aS} \cap \overline{bS}$. Take a net $\{s_\alpha\} \subset D$ such that $as_\alpha \to c$. For each α, we have $y \in s_\alpha K$. Choose $k_\alpha \in K$ such that $y = s_\alpha k_\alpha$. By passing to a subnet if necessary, we may assume $k_\alpha \to k_0$. From the joint continuity we have

$$ay = as_\alpha k_\alpha \to ck_0.$$

But $cS \subset \overline{bS}$. We have

$$cK = c\overline{Sx} \subset \overline{bSx}.$$

In particular, $ay = ck_0 \in \overline{bSx}$. Since $b \in D$ is arbitrary, it follows that $ay \in Y$ and hence $aY \subset Y$ ($a \in D$). This, in turn, implies that $sY \subset Y$ for all $s \in S$. Therefore $Y = K$ by the minimality of K. On the other hand, $Y \subset \overline{sSx} \subset sK$ for each $s \in S$. Thus (2) holds. □

Theorem 5.4. *Let S be a left reversible and metrizable semitopological semigroup. Then S has the fixed point property* (G^*). *In particular, $WAP(S) \cap LUC(S)$ has a LIM.*

Proof. From Lemma 5.2 S is strongly left reversible. Let $\{S_\alpha\colon\ \alpha \in I\}$ be the family of countable subsemigroups of S that satisfies the conditions (1)–(3) in the definition of the strong left reversibility. Each $\overline{S_\alpha}$ is separable. Let K be the weakly compact convex set described in the property (G^*). From Lemmas 5.3 and 3.3, any non-empty minimal $\overline{S_\alpha}$-invariant weakly compact subset F of K is Q-compact and hence is singleton as shown in the proof of Theorem 3.4. This shows that K contains a common fixed point for $\overline{S_\alpha}$. Now let $F_\alpha = \{k \in K\colon\ \overline{S_\alpha} k = k\}$. Then $\{F_\alpha\colon\ \alpha \in I\}$ is a family of non-empty weakly compact subsets of K that has the finite intersection property. So $\bigcap_{\alpha\in I} F_\alpha \neq \emptyset$ and, for $k \in \bigcap_{\alpha\in I} F_\alpha$, $S_\alpha k = k$ for all $\alpha \in I$. An element in $\bigcap_{\alpha\in I} F_\alpha$ serves as a common fixed point for $S = \bigcup_{\alpha\in I} S_\alpha$. Therefore ($\mathrm{G}^*$) holds. Moreover, if we denote $E = (WAP(S) \cap LUC(S))^*$ with the topology Q defined by the seminorms $\{p_f\colon\ f \in WAP(S) \cap LUC(S)\}$, where p_f is given by Eq. (2), and take $K =$ the set of all means on $WAP(S) \cap LUC(S)$. Then the action $s \mapsto \ell_s^*$ is weakly jointly continuous and Q-nonexpansive. So S has common fixed point in K, which is certainly a LIM on $WAP(S) \cap LUC(S)$. □

The next example shows that, in general, $WAP(S) \cap LUC(S)$ having a LIM does not imply $WAP(S)$ having a LIM. In other words, the fixed point property (F^*) does not imply the fixed point property (F).

Example 5.5. Let S be a semigroup such that $AP(S)$ has LIM, while $WAP(S)$ does not (an example of such a group is $S_{1,1}$). Let T be the spectrum of $WAP(S)$, the set of multiplicative means on $WAP(S)$. Equip T with the weak* topology. Then

(1) T is a compact semitopological semigroup,
(2) $C(T) = WAP(T) \cong WAP(S)$,
(3) $C(T) \cap LUC(T) = AP(T) \cong AP(S)$.

As a consequence, $WAP(T) \cap LUC(T)$ has a LIM but $WAP(T)$ does not, or equivalently, $C(T) \cap LUC(T)$ has a LIM but $C(T)$ does not.

Proof. Only the last assertion requires a proof. Indeed, if $f \in C(T) \cap LUC(T)$, the mapping $T \to C(T)$ specified by $t \mapsto \ell_t f$ is continuous, where $C(T)$ is equipped with the norm topology. Since T is compact, we have that $\mathcal{LO}(f) = \{\ell_t f\colon\ t \in T\}$ is compact. This shows $f \in AP(T)$. Conversely, if $f \in AP(T)$, then $\mathcal{LO}(f)$ is precompact in the norm topology. In particular, the topology of pointwise convergence and the norm topology agree on $\mathcal{LO}(f)$. If $t_\alpha \to t$, then $\ell_{t_\alpha} f \to \ell_t f$ pointwise. So $\|\ell_{t_\alpha} f - \ell_t f\| \to 0$. This shows that $f \in C(T) \cap LUC(T)$. Thus $C(T) \cap LUC(T) = AP(T)$. To show $AP(T) \cong AP(S)$, given $f \in AP(S) \subset WAP(S)$, we extend f to T and denote the extension by $\overline{f}$. Then $\overline{f} \in C(T)$. Moreover, $\{\ell_s \overline{f}\colon\ s \in S\}$ is precompact in the norm topology of $C(T)$. So $\overline{f} \in AP(T)$, i.e. $\overline{f} \in C(T) \cap LUC(T)$. On the other hand, if $\overline{f} \in C(T) \cap LUC(T)$, then $\overline{f}$ is the extension of some $f \in WAP(S)$. Since $\overline{f} \in AP(T)$, it follows that $f \in AP(S)$. □

The following diagram summarizes the relations among the fixed point properties discussed in this paper.

$$
\begin{array}{ccccccl}
\begin{array}{c}S \text{ is left reversible}\\ \&\ \text{metrizable}\end{array} & \Rightarrow & (G^*) & \Rightarrow & (F^*) & \overset{\mathrm{s}}{\Leftrightarrow} & \mathit{WAP}(S)\cap \mathit{LUC}(S) \text{ has LIM} \\
 & & \Uparrow & & \Uparrow\not\Downarrow & & \\
 & & (G) & \Rightarrow & (F) & \overset{\Rightarrow}{\not\Leftarrow} & (E) \overset{\mathrm{s}}{\Leftrightarrow} (E') \Leftrightarrow (D) \Leftrightarrow \mathit{AP}(S) \text{ has LIM} \\
 & & & & \mathrm{s}\Updownarrow & & \\
 & & & & \begin{array}{c}\mathit{WAP}(S)\\ \text{has LIM}\end{array} & &
\end{array}
$$

where "s" means the implication is under the condition that the semigroup is separable.

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